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LETTERS TO THE EDITOR

A NOTE ON THE NON-LINEAR PHENOMENA ACCOMPANYING PROPAGATION OF HIGH-FREQUENCY SOUND WAVES IN DUCTS

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1. INTRODUCTION

In the course of the last 20 years, the Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation [1,2]

$$\frac{\partial p}{\partial z}-\frac{c_0}{2}\int_{\tau_0}^{\tau}\Delta_\perp p\,\mathrm{d}\tau-\frac{\varepsilon}{\rho_0 c_0^3}(\tilde{p}-p_0)\frac{\partial p}{\partial \tau}-\frac{b}{2c_0^3}\frac{\partial^2 p}{\partial \tau^2}=0 \tag{1}$$

has widely been used to describe (mainly numerically) non-linear fields of beams of sound and of short pulses (see, for instance, Proceedings of the 10th ISNA 1984, 15th ISNA 1999).

Here, p represents the disturbance pressure of the gas medium, $p=\tilde{p}-p_0$, $\tilde{p}$ is the instantaneous value of the pressure, p_0, ρ_0 are the pressure and density of the undisturbed gas, respectively, c_0 is the speed of sound in the undisturbed gas, $\varepsilon=(\gamma+1)/2$, γ is the ratio of the specific heats, τ is the retarded time, $\tau=t-z/c_0$, z is the longitudinal co-ordinate. The quantity b is the effective diffusivity of sound which was introduced by Kirchhoff in 1868. The symbol $\Delta_\perp$ denotes the transversal part of the Laplace operator: $\Delta_\perp=\partial^2/\partial r^2+(1/r)\partial/\partial r$ in cylindrical co-ordinates in the presence of axial symmetry. The question of the value of τ_0 is determined by the condition that the value of τ_0 shall be equal to τ corresponding to the zero-value boundary conditions.

In what follows, the lossless version of equation (1) is considered as

$$\frac{\partial p}{\partial z}-\frac{c_0}{2}\int_{\tau_0}^{\tau}\Delta_\perp p\,\mathrm{d}\tau-\frac{\varepsilon}{\rho_0 c_0^3}(\tilde{p}-p_0)\frac{\partial p}{\partial \tau}=0 \tag{2}$$

attention being restricted to axisymmetric beams, with $p=p(r,z,\tau)$ in cylindrical co-ordinates. As shown by Rudenko and Soluyan [3], equation (2) is equivalent to the system of two coupled equations

$$\frac{\partial u}{\partial z}+\frac{1}{2}\mathrm{div}_\perp v-\frac{\varepsilon}{2c_0^2}\frac{\partial u^2}{\partial \tau}=0,\qquad \frac{\partial v}{\partial \tau}+c_0\frac{\partial u}{\partial \tau}=0. \tag{3, 4}$$

It has been used to describe, numerically by the finite difference method, non-linear wave propagation in tubes [4]. In the foregoing equations (3) and (4), $\mathrm{div}_\perp=\partial/\partial r+1/r$ in cylindrical co-ordinates, and u, v are the disturbance longitudinal and radial particle velocities respectively. The origin of equation (4) is the flow irrotationality condition. It is of

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interest to derive a relation between p and u which might be used to test numerical solutions to equation (2) and system (3, 4).

2. ANALYSIS

Such a relation accurate to the second order can easily be obtained from the equation of motion in the r direction, by using the equation of continuity, the equation of motion in the z direction, system (3, 4) and the equation of state for a fluid undergoing an isentropic process, namely

$$\rho = \tilde{\rho} - \rho_0 = (\tilde{p} - p_0)/c_0^2 + (\partial^2 \tilde{\rho}/\partial \tilde{p}^2)_s (\tilde{p} - p_0)^2/2. \quad (5)$$

Here, ρ is the density disturbance, $\tilde{\rho}$, $\tilde{p}$ are the instantaneous values of the density and pressure, respectively, and s is the entropy.

Indeed, in the presence of axial symmetry the continuity equation and the equation of motion in the r direction can be written in the physical co-ordinates r, z, t as

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial \tilde{\rho}\tilde{u}}{\partial z} + \frac{1}{r}\frac{\partial \tilde{\rho}\tilde{v}r}{\partial r} = 0, \qquad \tilde{\rho}\frac{\partial \tilde{v}}{\partial t} + \tilde{\rho}\tilde{v}\frac{\partial \tilde{v}}{\partial r} + \tilde{\rho}\tilde{u}\frac{\partial \tilde{v}}{\partial z} + \frac{\partial \tilde{p}}{\partial r} = 0, \quad (6, 7)$$

where $\tilde{u}$, $\tilde{v}$ are the instantaneous values of longitudinal and transversal particle velocities respectively. It should be noted that, since the KZK equation is formulated by assuming zero mean flow, the instantaneous values of longitudinal and radial particle velocities must be equal to the corresponding disturbance velocities, $\tilde{u} = u$, $\tilde{v} = v$.

From equations (6) and (7),

$$\partial \tilde{\rho}\tilde{v}/\partial t + (1/r)\partial r \tilde{\rho}\tilde{v}^2/\partial r + \partial \tilde{\rho}\tilde{u}\tilde{v}/\partial z + \partial \tilde{p}/\partial r = 0. \quad (8)$$

Substitution of the equation of state (5) into equation (8) yields, in the second order non-linear-acoustics approximation, the equation

$$\rho_0 \partial v/\partial t + (\rho_0/c_0)\, \partial uv/\partial t + \rho_0 \partial uv/\partial z + (\rho_0/r)\partial r v^2/\partial r + \partial p/\partial r = 0. \quad (9)$$

After transposing from the physical co-ordinates z, t to the co-ordinates z, τ, equation (9) takes the form

$$\rho_0 \partial v/\partial \tau + \rho_0 \partial uv/\partial z + (\rho_0/r)\, \partial r v^2/\partial r + \partial p/\partial r = 0. \quad (10)$$

Use of the fact, see references [1–4], that

$$u = u(\tau, \mu z, \sqrt{\mu} r), \quad v = v(\tau, \mu z, \sqrt{\mu} r), \quad v/c_0 \sim \mu^{3/2}, \quad (11)$$

where $\mu \sim u/c_0$ is the dimensionless wave amplitude, $\mu \ll 1$, into equation (10) yields, in the second order approximation, the equation

$$\rho_0 \partial v/\partial \tau + \partial p/\partial r = 0. \quad (12)$$

It is worth noting that it follows from the paraxial approximation (11) that

$$\partial/\partial \tau = O(1), \qquad \partial/\partial z = O(\mu), \qquad \partial/\partial r = O(\mu^{1/2}).$$

The exact expression of the flow irrotationality condition can be written as

$$\partial u/\partial r - \partial v/\partial z + (1/c_0)\partial v/\partial \tau = 0.$$

Now on making use of the paraxial approximation, one finds that this equation can be written as

$$\frac{\partial u}{\partial r} = O(\mu^{3/2}) = -\frac{1}{c_0}\frac{\partial v}{\partial \tau}\left(1 - \frac{\partial v/\partial z}{(1/c_0)\partial v/\partial \tau}\right) = -\frac{1}{c_0}\frac{\partial v}{\partial \tau}(1 - O(\mu)).$$

It follows from the latter equation that

$$\frac{\partial v}{\partial \tau} = -c_0\frac{O(\mu^{3/2})}{(1 - O(\mu))} \approx -c_0 O(\mu^{3/2})[1 + O(\mu) + O(\mu^2) + \cdots].$$

It is clear from this equation that $v/c_0 \approx O(\mu^{3/2})$. This provides a proof of the validity of the ratio v/u (11). Substitution of equation (4) into equation (12), leads to the relation

$$\partial p/\partial r - \rho_0 c_0 \partial u/\partial r = 0. \tag{13}$$

The right-hand side error of equation (13) is $O(\mu^{5/2})$, where O is the usual order symbol. Thus, by integrating equation (13), one readily obtains

$$p(z, r, \tau) - \rho_0 c_0 u(z, r, \tau) = f(z, r, \tau), \tag{14}$$

where the function $f(z, r, \tau)$ is of order μ^2.

It should be noted here that equation (14) can be combined with equations (3, 4) to obtain equation (2) or with equations (2, 4) to obtain equation (3). Indeed, by introducing expression for u from equation (14) into equations (3, 4) one obtains

$$\begin{aligned} &\frac{1}{\rho_0 c_0}\frac{\partial p}{\partial z} - \frac{1}{\rho_0 c_0}\frac{\partial f}{\partial z} + \frac{1}{2}\mathrm{div}_\perp v - \frac{\varepsilon}{2c_0^2}\frac{1}{\rho_0^2 c_0^2}\frac{\partial (p-f)^2}{\partial \tau} \\ &= \frac{1}{\rho_0 c_0}\frac{\partial p}{\partial z} + \tfrac{1}{2}\mathrm{div}_\perp v - \frac{\varepsilon}{2c_0^2}\frac{1}{\rho_0^2 c_0^2}\frac{\partial p^2}{\partial \tau} \\ &\quad - \frac{1}{\rho_0 c_0}\frac{\partial f}{\partial z} + \frac{\varepsilon}{c_0^2}\frac{1}{\rho_0^2 c_0^2}\frac{\partial pf}{\partial \tau} - \frac{\varepsilon}{2c_0^2}\frac{1}{\rho_0^2 c_0^2}\frac{\partial f^2}{\partial \tau} = 0, \end{aligned} \tag{15}$$

$$(\partial v/\partial \tau) + c_0(1/\rho_0 c_0)\,\partial p/\partial r - c_0(1/\rho_0 c_0)\partial f/\partial r = 0. \tag{16}$$

The sum of the last three terms in equation (15) is of magnitude $O(\mu^3)$ and the last term in equation (16) is of magnitude $O(\mu^{5/2})$ and thus, with the accuracy of this analysis, are neglected to give

$$(1/\rho_0 c_0)\partial p/\partial z + \tfrac{1}{2}\mathrm{div}_\perp v - (\varepsilon/2c_0^2)(1/\rho_0^2 c_0^2)\partial p^2/\partial \tau = 0, \tag{17}$$

$$\partial v/\partial \tau + (1/\rho_0)\partial p/\partial r = 0. \tag{18}$$

Integration of equation (18) with respect to τ from $\tau = \tau_0$ to the current τ, yields the relation between v and p:

$$v(z, r, \tau) = v(z, r, \tau_0) - \frac{1}{\rho_0} \int_{\tau_0}^{\tau} \frac{\partial p}{\partial r} \, \mathrm{d}\tau. \tag{19}$$

where $v(z, r, \tau_0)$ is equal to zero.

Substitution of relation (19) into equation (17) yields equation (2).

Following a similar procedure, the expression for p from equation (14) can be introduced into equation (2):

$$\rho_0 c_0 \frac{\partial u}{\partial z} + \frac{\partial f}{\partial z} - \frac{c_0}{2} \int_{\tau_0}^{\tau} \Delta_\perp (\rho_0 c_0 u + f) \, \mathrm{d}\tau - \frac{\varepsilon}{\rho_0 c_0^3} (\rho_0 c_0 u + f) \frac{\partial}{\partial \tau} (\rho_0 c_0 u + f)$$

$$= \rho_0 c_0 \frac{\partial u}{\partial z} + \frac{\partial f}{\partial z} - \frac{c_0}{2} \int_{\tau_0}^{\tau} \left(\rho_0 c_0 \frac{\partial^2 u}{\partial r^2} + \frac{\rho_0 c_0}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} \right) \mathrm{d}\tau$$

$$- \frac{\varepsilon}{2 \rho_0 \mathrm{c}_0^3} \frac{\partial}{\partial \tau} (\rho_0^2 c_0^2 u^2 + 2 \rho_0 c_0 u f + f^2)$$

$$= \rho_0 c_0 \frac{\partial u}{\partial z} - \frac{\rho_0 c_0^2}{2} \int_{\tau_0}^{\tau} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \mathrm{d}\tau - \frac{\varepsilon \rho_0}{2 c_0} \frac{\partial u^2}{\partial \tau}$$

$$- \frac{c_0}{2} \int_{\tau_0}^{\tau} \left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} \right) \mathrm{d}\tau + \frac{\partial f}{\partial z} - \frac{\varepsilon}{c_0^2} \frac{\partial}{\partial \tau} (uf) - \frac{\varepsilon}{2 \rho_0 c_0^3} \frac{\partial f^2}{\partial \tau} = 0. \tag{20}$$

The sum of the last four terms in equation (20) is of magnitude $O(\mu^{5/2})$ and thus, within the accuracy of this analysis, is neglected to give

$$\rho_0 c_0 \frac{\partial u}{\partial} - \frac{\rho_0 c_0^2}{2} \int_{\tau_0}^{\tau} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \mathrm{d}\tau - \frac{\varepsilon \rho_0}{2 c_0} \frac{\partial u^2}{\partial \tau} = 0. \tag{21}$$

After multiplying by r, differentiating with respect to r and multiplying by $1/r$, equation (4) becomes

$$(\partial/\partial\tau)((1/r)\partial rv/\partial r) = -\,(c_0/r)(\partial/\partial r)\,(r\partial u/\partial r). \tag{22}$$

By introducing the expression for $(1/r)\partial/\partial r\ (r\partial u/\partial r)$ from equation (22) in the second term of equation (21) one obtains

$$\rho_0 c_0 \frac{\partial u}{\partial z} + \frac{\rho_0 c_0^2}{2} \frac{1}{c_0} \int_{\tau_0}^{\tau} \frac{\partial}{\partial \tau} \left(\frac{1}{r} \frac{\partial rv}{\partial r} \right) \mathrm{d}\tau - \frac{\varepsilon \rho_0}{2 c_0} \frac{\partial u^2}{\partial \tau}$$

$$= \rho_0 c_0 \left[\frac{\partial u}{\partial z} + \frac{1}{2} \int_{\tau_0}^{\tau} \frac{\partial}{\partial \tau} \left(\frac{1}{r} \frac{\partial rv}{\partial r} \right) \mathrm{d}\tau - \frac{\varepsilon}{2 c_0^2} \frac{\partial u^2}{\partial \tau} \right] = 0. \tag{23}$$

This equation may be written as

$$\frac{\partial u}{\partial z}+\frac{1}{2}\int_{\tau_0}^{\tau}\frac{\partial}{\partial \tau}\left(\frac{1}{r}\frac{\partial rv}{\partial r}\right)\mathrm{d}\tau-\frac{\varepsilon}{2c_0^2}\frac{\partial u^2}{\partial \tau}=0. \tag{24}$$

Integration of the second term in equation (24) from $\tau=\tau_0=-\infty$ to the current τ and use of the zero-value boundary conditions at $\tau=\tau_0=-\infty$ leads to equation (3). These prove the assertion that the KZK equation (2) is equivalent to system (3, 4). The equation of motion in the z direction in the second order approximation may be written as

$$\rho_0\,\partial u/\partial\tau+\partial p/\partial z-(1/c_0)\partial p/\partial\tau=0. \tag{25}$$

The right-hand side error of equation (25) is $O(\mu^3)$. After inserting equation (14) into equation (25), the latter takes the form

$$\rho_0 c_0\,\partial u/\partial z+\partial f(z,r,\tau)/\partial z-(1/c_0)\partial f(z,r,\tau)/\partial\tau=0. \tag{26}$$

The second term in equation (26) is of magnitude $O(\mu^3)$. Hence, it is omitted in the derivation to the second order. The result is written as

$$\rho_0 c_0\,\partial u/\partial z-(1/c_0)\partial f(z,r,\tau)/\partial\tau=0. \tag{27}$$

When space-averaging over the tube cross-section, equations (3), (14) and (27) are reduced to the following equations respectively:

$$\partial\bar{u}/\partial z-(\varepsilon/2c_0^2)\partial\bar{u}^2/\partial\tau=0,\qquad \overline{p(z,\tau)}-\rho_0 c_0\overline{u(z,\tau)}=\overline{f(z,\tau)}: \tag{28, 29}$$

$$\rho_0 c_0\;\partial\bar{u}/\partial z-(1/c_0)\partial\overline{f(z,\tau)}/\partial\tau=0, \tag{30}$$

where

$$\overline{u(z,\tau)}=\frac{2}{a^2}\int_0^a ru(z,r,\tau)\,\mathrm{d}r,\quad \overline{u^2(z,\tau)}=\frac{2}{a^2}\int_0^a ru^2(z,r,\tau)\,\mathrm{d}r, \tag{31, 32}$$

$$\overline{p(z,\tau)}=\frac{2}{a^2}\int_0^a rp(z,r,\tau)\,\mathrm{d}r,\quad \overline{f(z,\tau)}=\frac{2}{a^2}\int_0^a rf(z,r,\tau)\,\mathrm{d}r. \tag{33, 34}$$

In deriving equation (28), the boundary condition (see reference [4]), $v(z,r,\tau)=0$ at $r=a$ is used. From equations (28)–(30), one has

$$\partial\bar{u}/\partial\tau+(\varepsilon/2c_0)\partial\overline{u^2}/\partial\tau-(1/\rho_0 c_0)\partial\bar{p}/\partial\tau=0. \tag{35}$$

Integration of equation (35) with respect to τ from $\tau=\tau_0=-\infty$ to the current τ and use of the zero-value boundary conditions at $\tau=\tau_0=-\infty$, yields the following relation between p and u:

$$\overline{p(z,\tau)}=\rho_0 c_0\overline{u(z,\tau)}+(\rho_0\varepsilon/2)\overline{u^2(z,r)}. \tag{36}$$

3. DISCUSSION AND CONCLUSIONS

Since the sum of the neglected terms in the original equation (10) is of magnitude $O(\mu^{7/2})$, it follows that the right-hand side error of equation (36) is $O(\mu^3)$.

In order to investigate the validity of the obtained formula (36), one compares this formula with the one-dimensional Riemann solution for a lossless compressible fluid with an adiabatic (barotropic) equation of state $\tilde{p} = p_0(\tilde{\rho}/\rho_0)^{\gamma}$:

$$\begin{aligned}\tilde{p} &= p_0\left(1 + \frac{\gamma-1}{2}\frac{u}{c_0}\right)^{2\gamma/\gamma-1} \\ &= p_0\left[1 + \frac{\gamma-1}{2}\frac{2\gamma}{\gamma-1}\frac{u}{c_0} + \frac{(\gamma-1)^2}{4}\left(\frac{2\gamma}{\gamma-1} - 1\right)\frac{2\gamma}{(\gamma-1)}\frac{1}{2}\frac{u^2}{c_0^2} + O\left(\left(\frac{u}{c_0}\right)^3\right)\right] \\ &= p_0 + \gamma\frac{p_0}{c_0}u + \frac{(\gamma-1)^2(2\gamma-\gamma+1)}{8(\gamma-1)}\frac{2}{(\gamma-1)}\gamma\frac{p_0}{c_0^2}u^2 + O(\mu^3) \\ &\approx p_0 + \rho_0 c_0 u + \frac{1}{2}\frac{(\gamma+1)}{2}\rho_0 u^2 = p_0 + \rho_0 c_0 u + \frac{\varepsilon}{2}\rho_0 u^2,\end{aligned} \tag{37}$$

where $\tilde{\rho}$ is the instantaneous value of the density.

In deriving equation (37), $c_0^2 : c_0^2 = \gamma p_0/\rho_0$ and terms of higher order in $(u/c_0)(u/c_0 \sim \mu, \mu \ll 1)$ have been neglected, consistent with the quadratic form of expression (5). Formula (37) can be rewritten as

$$\tilde{p} - p_0 = p = \rho_0 c_0 u + (\varepsilon/2)\rho_0 u^2. \tag{38}$$

It can be easily seen that when space-averaging over the tube cross-section, equation (38) is reduced to equation (36), thus validating the present mathematical solution.

Finally, it is important to notice that by applying the preceding space-averaging procedure over a whole transversal plane in the case of the free field in the presence of axial symmetry with the corresponding zero-value boundary conditions at $r \to \infty$, one can obtain the "impedance relation" connecting particle velocity and sound pressure in a quasi-one-dimensional non-linear progressive wave, which is the same as that obtained here for ducts. Thus, equation (36) is not necessarily limited to sound in ducts.

Formula (36) would be useful in the testing of solutions to equation (2) and system (3, 4) obtained by various numerical methods.

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